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On the derivative of the Legendre function of the first kind with respect to its degree

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Abstract

We study the derivative of the Legendre function of the first kind, $P_\nu(z)$, with respect to its degree ν. At first, we provide two contour integral representations for $\partial P_\nu(z)/\partial\nu$. Then, we proceed to investigate the case of $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$, with n being an integer; this case is met in some physical and engineering problems. Since it holds that $[\partial P_{\nu'}(z)/\partial\nu']_{\nu'=-\nu-1} = -[\partial P_{\nu'}(z)/\partial\nu']_{\nu'=\nu}$, we focus on the sub-case of n being a non-negative integer. We show that

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n} = P_n(z)\ln\frac{z+1}{2} + R_n(z) \qquad (n \in \mathbb{N})$$

where $R_n(z)$ is a polynomial in z of degree n. We present alternative derivations of several known explicit expressions for $R_n(z)$ and also add some new. A generating function for $R_n(z)$ is also constructed. Properties of the polynomials $V_n(z) = [R_n(z) + (-1)^n R_n(-z)]/2$ and $W_{n-1}(z) = -[R_n(z)-(-1)^n R_n(-z)]/2$ are also investigated. It is found that $W_{n-1}(z)$ is the Christoffel polynomial, well known from the theory of the Legendre function of the second kind, $Q_n(z)$. As examples of applications of the results obtained, we present non-standard derivations of some representations of $Q_n(z)$, sum to closed forms some Legendre series, evaluate some definite integrals involving Legendre polynomials and also derive an explicit representation of the indefinite integral of the Legendre polynomial squared.

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1. Introduction

The importance of the Legendre functions for applied mathematics, for mathematical and theoretical physics, and also for theoretical engineering, is comparable with that of the Bessel functions. This explains why over the past two centuries the Legendre functions have been objects of extensive studies. Partial results of these investigations have been summarized in handbooks of special functions [1–5] and in several monographs [6–17]. The research on the Legendre functions is still going on, being motivated both by pure scientific interest as well as by particular problems arising in applications of these functions in the aforementioned disciplines.

The present paper is devoted to the study of the derivative of the Legendre function of the first kind, $P_\nu(z)$, with respect to its degree ν. Primarily, we shall be interested in the family of functions $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$, $n \in \mathbb{N}$, which are met in solving some boundary value problems of potential theory [18], of heat conduction in solids [19] and of electromagnetism [20], in the general theory of relativity [21, 22], in the theory of angular momentum eigenfunctions for spin one-half particles [23] and also in the theory of Green's functions for the spherical Helmholtz operator [24]. The direct motivation to undertake the research reported in this paper has come from our, somewhat surprising, finding that in the literature there seems to be no work aimed at a comprehensive and systematic investigation of properties of $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$; the present work purports to fill in this gap.

The structure of the paper is as follows. In section 2, we give a brief overview of results obtained by other authors in their fragmentary investigations on $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$. Section 3 is devoted to recalling some basic facts from the theory of the Legendre function of the first kind, to be used in later parts of the work. In section 4, we provide two contour integral representations of $\partial P_\nu(z)/\partial\nu$; for completeness, a known series representation of $\partial P_\nu(z)/\partial\nu$ is also given there. In addition, some useful functional relations obeyed by this function are listed. In section 5, we focus on $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ and prove that it may be written as

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n} = P_n(z)\ln\frac{z+1}{2} + R_n(z) \qquad (n \in \mathbb{N}) \tag{1.1}$$

where $P_n(z)$ is the Legendre polynomial, while $R_n(z)$ is a polynomial in z of degree n. The next parts of section 5 are devoted to the study of properties of $R_n(z)$. In particular, we construct a generating function for $R_n(z)$, rederive in alternative ways several known representations of this polynomial and also add some new. Section 5 ends with a brief investigation of the related polynomials $V_n(z) = [R_n(z)+(-1)^n R_n(-z)]/2$ and $W_{n-1}(z) = -[R_n(z)-(-1)^n R_n(-z)]/2$; it is found that the latter are the Christoffel polynomials, well known from the theory of the Legendre function of the second kind, $Q_n(z)$. The final section 6 shows some exemplary applications of the results obtained in section 5. Specifically, we present non-standard derivations of some properties of the Legendre function of the second kind, then sum to closed forms some Legendre series and evaluate two definite integrals involving the Legendre polynomials; also, we derive a formula for the indefinite integral of the Legendre polynomial squared.

Throughout the whole paper, unless otherwise stated, it will be implicit that $\nu \in \mathbb{C}$, $n \in \mathbb{N}$, $z \in \mathbb{C}$ and $-1 \leqslant x \leqslant 1$.

2. An overview of research done on $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$

The earliest investigations on $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ we have been able to trace were presented in two, nowadays forgotten, papers by Bromwich [18] and Jolliffe [25], published in the second decade of the 20th century. In the first of these works [18], concerning some particular boundary value problems of potential theory, Bromwich obtained formula (1.1) and gave the following two representations of the polynomial $R_n(z)$:

$$R_n(z) = \sum_{k=1}^{n}\frac{1}{k}[P_k(z) - P_{k-1}(z)]P_{n-k}(z) \tag{2.1}$$

$$R_n(z) = 2\sum_{k=0}^{n-1}(-1)^{n+k}\frac{2k+1}{(n-k)(n+k+1)}[P_k(z) - P_n(z)]. \tag{2.2}$$

In the short note [25], Jolliffe proved that $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ may be represented as

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n} = \frac{1}{2^{n-1}n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z^2-1)^n \ln\frac{z+1}{2}\right] - P_n(z)\ln\frac{z+1}{2}. \tag{2.3}$$

A compact expression for $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ was provided by Hobson at the bottom of p 172 in the classic monograph [11], but it appears to be incorrect.

Somewhat later, in the study [20] (cf also [26]) on the electromagnetic antenna theory, Schelkunoff made an ingenious use of the Hille's formula [15, p 68], linking $P_\nu(z)$ and $P_\nu(-z)$, to obtain the result (1.1) with the polynomial $R_n(z)$ expressed as[1]

$$R_n(z) = 2\sum_{k=1}^{n}\frac{(n+k)!}{(k!)^2(n-k)!}[\psi(n+k+1) - \psi(n+1)]\left(\frac{z-1}{2}\right)^k \tag{2.4}$$

where

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta)}\frac{\mathrm{d}\Gamma(\zeta)}{\mathrm{d}\zeta} \tag{2.5}$$

is the digamma function.

The more general problem of evaluating the derivative $\left[\partial P_\nu^{(\mu)}(z)/\partial\nu\right]_{\nu=n}$, where $P_\nu^{(\mu)}(z)$ is the associated Legendre function of the first kind, was investigated by Robin in [27] and in [15, section 75]; however, his result looks fairly complicated.

The functions $\left[\partial P_\nu^{(m)}(z)/\partial\nu\right]_{\nu=n}$, with $m \in \mathbb{N}$, were studied independently by Tsu [28] and Carlson [29]. For $m = 0$, the finding of Carlson reduces to formula (1.1) with $R_n(z)$ having the same form as in the Schelkunoff's expression (2.4). Tsu also arrived at equation (1.1), but being unable to derive an explicit representation of $R_n(z)$, instead he provided a recurrence relation obeyed by these polynomials. With the aid of that recurrence, augmented by the initial condition $R_0(z) = 0$, in principle one may find any of the polynomials $R_n(z)$.

Essentially, the same results as those of Tsu [28] were obtained for $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ by Hoenselaers [21] and by the present author [23, appendix A].

The most recent investigation on $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ seems to be that of the present author [24], who, being at that time unaware of the work of Bromwich [18], rederived his results (1.1) and (2.2).

It is noteworthy that in many cases authors of the aforecited articles apparently were not aware of results obtained by other researchers. The exceptions were Jolliffe [25] and Hobson [11, section 112], who both cited the Bromwich's paper [18], Robin [15, 27], who referred to the paper of Schelkunoff [20], and Carlson [29], who mentioned the Schelkunoff's work [20] and also a relevant section in Robin's monograph [15].

3. The Legendre function $P_\nu(z)$

It is well known [30, section 15.2] that the Legendre function of the first kind, $P_\nu(z)$, may be defined as the Schläfli contour integral:

$$P_\nu(z) = \frac{1}{2^{\nu+1}\pi\mathrm{i}}\oint_{C^{(+)}}\mathrm{d}t\,\frac{(t^2-1)^\nu}{(t-z)^{\nu+1}}. \tag{3.1}$$

[1] The reader should be warned that Schelkunoff [20] and Robin [14–16] defined the digamma function as

$$\psi(\zeta) = \frac{1}{\Gamma(\zeta+1)}\frac{\mathrm{d}\Gamma(\zeta+1)}{\mathrm{d}\zeta}$$

rather than as in our equation (2.5). For this reason, Schelkunoff's result for $R_n(z)$, quoted also by Robin in [15, equation (331 *ter*)], seemingly differs from our equation (2.4).

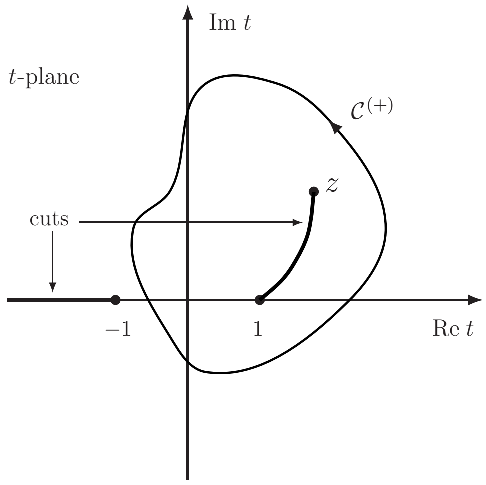

Figure 1. The complex t-plane and the integration contour $\mathcal{C}^{(+)}$ for the definition (3.1) of the Legendre function $P_\nu(z)$. For the sake of applications presented in section 4.1, the cut joining the points $t = 1$ and $t = z$ (being two out of four branch points of the integrand in equation (3.1)) has been chosen in the form (3.2).

If ν is not an integer, the integrand in equation (3.1) has four branch points, located at $t = \pm 1, t = z$ and $|t| = \infty$, and cuts in the complex t-plane are necessary to make the integrand single valued (cf figure 1). Following the common convention [30, chapter 15], one of these cuts will be chosen here as the semi-line along the real axis from $t = -1$ to $t = -\infty$. In view of later applications presented in section 4.1, we choose the second cut in the form of the curve joining the points $t = 1$ and $t = z$, parametrized as

$$t = \frac{1 + \eta z}{z + \eta} \qquad (1 \leqslant \eta < \infty). \tag{3.2}$$

The integration contour $\mathcal{C}^{(+)}$ is a closed curve enclosing the points $t = 1$ and $t = z$, such that it does not cross either of the two cuts and is run counterclockwise. In addition, at the point on the right of the point $t = 1$ (and on the right of z if z be real), where the contour $\mathcal{C}^{(+)}$ crosses the real axis, one sets $\arg(t \pm 1) = 0$ and $|\arg(t - z)| < \pi$. The function $P_\nu(z)$ defined by equation (3.1) is single valued and analytic throughout the whole complex z-plane cut along the real axis from $z = -1$ to $z = -\infty$. If ν is an integer, the cuts both in the t-plane and z-plane are unnecessary.

If $|z - 1| < 2$, an alternative representation of $P_\nu(z)$, in terms of the Gauss hypergeometric series, is

$$P_\nu(z) = {}_2F_1\left(-\nu, \nu + 1; 1; \frac{1 - z}{2}\right) = \sum_{k=0}^{\infty} \frac{(-\nu)_k (\nu + 1)_k}{(k!)^2} \left(\frac{1 - z}{2}\right)^k \qquad (|z - 1| < 2) \tag{3.3}$$

where

$$(\zeta)_k = \frac{\Gamma(\zeta + k)}{\Gamma(\zeta)} \tag{3.4}$$

is the Pochhammer symbol [4].

If $\nu = n \in \mathbb{N}$, the function $P_\nu(z)$ degenerates to the Legendre polynomial

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n (z^2-1)^n}{\mathrm{d}z^n}. \tag{3.5}$$

The following well-known properties of the function $P_\nu(z)$ may be derived from the definition (3.1):

$$P_\nu(1) = 1 \tag{3.6}$$

$$P_{-\nu-1}(z) = P_\nu(z) \tag{3.7}$$

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + \nu(\nu+1)\right] P_\nu(z) = 0 \tag{3.8}$$

$$(\nu+1)P_{\nu+1}(z) - (2\nu+1)zP_\nu(z) + \nu P_{\nu-1}(z) = 0 \tag{3.9}$$

$$\frac{\mathrm{d}P_{\nu+1}(z)}{\mathrm{d}z} - z\frac{\mathrm{d}P_\nu(z)}{\mathrm{d}z} = (\nu+1)P_\nu(z) \tag{3.10}$$

$$z\frac{\mathrm{d}P_\nu(z)}{\mathrm{d}z} - \frac{\mathrm{d}P_{\nu-1}(z)}{\mathrm{d}z} = \nu P_\nu(z) \tag{3.11}$$

$$\frac{\mathrm{d}P_{\nu+1}(z)}{\mathrm{d}z} - \frac{\mathrm{d}P_{\nu-1}(z)}{\mathrm{d}z} = (2\nu+1)P_\nu(z) \tag{3.12}$$

$$(z+1)\frac{\mathrm{d}P_\nu(z)}{\mathrm{d}z} - (z+1)\frac{\mathrm{d}P_{\nu-1}(z)}{\mathrm{d}z} = \nu P_\nu(z) + \nu P_{\nu-1}(z) \tag{3.13}$$

$$(z^2-1)\frac{\mathrm{d}P_\nu(z)}{\mathrm{d}z} = \nu z P_\nu(z) - \nu P_{\nu-1}(z). \tag{3.14}$$

We shall make extensive use of these properties in the following sections.

4. Representations and some functional properties of $\partial P_\nu(z)/\partial\nu$

4.1. Contour integral representations of $\partial P_\nu(z)/\partial\nu$

If we differentiate equation (3.1) with respect to ν, after some simple manipulations we obtain the first of two contour integral representations of the derivative $\partial P_\nu(z)/\partial\nu$ given in this section:

$$\frac{\partial P_\nu(z)}{\partial\nu} = \frac{1}{2^{\nu+1}\pi\mathrm{i}} \oint_{C^{(+)}} \mathrm{d}t\, \frac{(t^2-1)^\nu}{(t-z)^{\nu+1}} \ln\frac{t^2-1}{2(t-z)}. \tag{4.1}$$

The cuts in the complex t-plane and the integration contour are identical as in the case of the integral (3.1) defining the function $P_\nu(z)$, and the principal branch of the logarithm is used. It is seen from the above equation that the function $\partial P_\nu(z)/\partial\nu$ is single valued and analytic throughout the whole complex z-plane cut along the real axis from $z = -1$ to $z = -\infty$.

Before proceeding further, let us consider the transformation

$$\tau = \frac{1-zt}{z-t} = -1 + (z+1)\frac{t-1}{t-z}. \tag{4.2}$$

It maps, in a one-to-one way, the complex t-plane onto the complex τ-plane. In particular, the points $t = -1, +1, z$ and ∞ are mapped onto the points $\tau = +1, -1, \infty$ and z, respectively, the cut $t = -\eta$ $(1 \leqslant \eta < \infty)$ is mapped onto the cut

$$\tau = \frac{1+z\eta}{z+\eta} \qquad (1 \leqslant \eta < \infty) \tag{4.3}$$

(cf equation (3.2)) and the cut (3.2) is mapped onto the cut

$$\tau = -\eta \qquad (1 \leqslant \eta < \infty). \tag{4.4}$$

Furthermore, the t-contour $\mathcal{C}^{(+)}$ generates the τ-contour $\mathcal{D}^{(-)}$, enclosing the points $\tau = 1$ and $\tau = z$, run clockwise, and not crossing the cuts (4.3) and (4.4).

Now, to obtain the second contour integral representation of the derivative $\partial P_\nu(z)/\partial\nu$, we rewrite equation (4.1) as

$$\frac{\partial P_\nu(z)}{\partial\nu} = I_\nu^{(1)}(z) + I_\nu^{(2)}(z) \tag{4.5}$$

where

$$I_\nu^{(1)}(z) = \frac{1}{2^{\nu+1}\pi\mathrm{i}} \oint_{\mathcal{C}^{(+)}} \mathrm{d}t\, \frac{(t^2-1)^\nu}{(t-z)^{\nu+1}} \ln\frac{t+1}{2} \tag{4.6}$$

and

$$I_\nu^{(2)}(z) = \frac{1}{2^{\nu+1}\pi\mathrm{i}} \oint_{\mathcal{C}^{(+)}} \mathrm{d}t\, \frac{(t^2-1)^\nu}{(t-z)^{\nu+1}} \ln\frac{t-1}{t-z}. \tag{4.7}$$

Applying the variable transformation (4.2) to the integral (4.7) gives

$$I_\nu^{(2)}(z) = -\frac{1}{2^{\nu+1}\pi\mathrm{i}} \oint_{\mathcal{D}^{(-)}} \mathrm{d}\tau\, \frac{(\tau^2-1)^\nu}{(\tau-z)^{\nu+1}} \ln\frac{\tau+1}{z+1}. \tag{4.8}$$

It is evident that the contour $\mathcal{D}^{(-)}$ in equation (4.8) may be deformed into the contour $\mathcal{C}^{(-)}$, which is a negatively oriented contour $\mathcal{C}^{(+)}$, without changing the value of the integral. Then, switching from $\mathcal{C}^{(-)}$ to $\mathcal{C}^{(+)}$, changing *the name* of the integration variable from τ to t and inserting the result, together with equation (4.6), into equation (4.5) leads to the following contour integral representation of $\partial P_\nu(z)/\partial\nu$:

$$\frac{\partial P_\nu(z)}{\partial\nu} = \frac{1}{2^{\nu+1}\pi\mathrm{i}} \oint_{\mathcal{C}^{(+)}} \mathrm{d}t\, \frac{(t^2-1)^\nu}{(t-z)^{\nu+1}} \ln\frac{(t+1)^2}{2(z+1)} \tag{4.9}$$

alternative to that in equation (4.1). Equation (4.9) may be rewritten in the form

$$\frac{\partial P_\nu(z)}{\partial\nu} = \frac{1}{2^{\nu}\pi\mathrm{i}} \oint_{\mathcal{C}^{(+)}} \mathrm{d}t\, \frac{(t^2-1)^\nu}{(t-z)^{\nu+1}} \ln\frac{t+1}{2} - P_\nu(z)\ln\frac{z+1}{2} \tag{4.10}$$

being the consequence of the definition (4.1).

4.2. A series representation of $\partial P_\nu(z)/\partial\nu$

Differentiating equation (3.3) with respect to ν, after exploiting the fact that

$$\frac{\mathrm{d}(\zeta)_k}{\mathrm{d}\zeta} = (\zeta)_k[\psi(\zeta+k) - \psi(\zeta)] \tag{4.11}$$

where $\psi(\zeta)$ is the digamma function (2.5), we obtain $\partial P_\nu(z)/\partial\nu$ in the form [24, 28]

$$\frac{\partial P_\nu(z)}{\partial\nu} = \sum_{k=1}^{\infty} \frac{(-\nu)_k(\nu+1)_k}{(k!)^2}[\psi(\nu+k+1) - \psi(\nu+1) + \psi(-\nu) - \psi(-\nu+k)]\left(\frac{1-z}{2}\right)^k \qquad (|z-1|<2). \tag{4.12}$$

Since it is easily provable that

$$-\psi(\nu+1) + \psi(-\nu) - \psi(-\nu+k) = -\psi(\nu-k+1) \tag{4.13}$$

equation (4.12) may be simplified, becoming [23, 24, 27]

$$\frac{\partial P_\nu(z)}{\partial\nu} = \sum_{k=1}^{\infty} \frac{(-\nu)_k(\nu+1)_k}{(k!)^2}[\psi(\nu+k+1) - \psi(\nu-k+1)]\left(\frac{1-z}{2}\right)^k \qquad (|z-1|<2). \tag{4.14}$$

4.3. Some functional properties of $\partial P_\nu(z)/\partial\nu$

Various functional relations obeyed by $\partial P_\nu(z)/\partial\nu$ may be derived from the representations (4.1) or (4.9). However, in many cases it will be much easier to obtain them by differentiating the known relations obeyed by $P_\nu(z)$. Proceeding in this way, from equations (3.6) to (3.8) we deduce the property

$$\frac{\partial P_\nu(1)}{\partial\nu} = 0, \tag{4.15}$$

the symmetry relation

$$\left.\frac{\partial P_{\nu'}(z)}{\partial\nu'}\right|_{\nu'=-\nu-1} = -\left.\frac{\partial P_{\nu'}(z)}{\partial\nu'}\right|_{\nu'=\nu}, \tag{4.16}$$

and the differential relation

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + \nu(\nu+1)\right]\frac{\partial P_\nu(z)}{\partial\nu} = -(2\nu+1)P_\nu(z). \tag{4.17}$$

Furthermore, from equations (3.9) to (3.14) we infer the inhomogeneous three-term recurrence relation

$$(\nu+1)\frac{\partial P_{\nu+1}(z)}{\partial\nu} - (2\nu+1)z\frac{\partial P_\nu(z)}{\partial\nu} + \nu\frac{\partial P_{\nu-1}(z)}{\partial\nu} = -P_{\nu+1}(z) + 2zP_\nu(z) - P_{\nu-1}(z) \tag{4.18}$$

which, again in virtue of equation (3.9), may also be rewritten as

$$(\nu+1)\frac{\partial P_{\nu+1}(z)}{\partial\nu} - (2\nu+1)z\frac{\partial P_\nu(z)}{\partial\nu} + \nu\frac{\partial P_{\nu-1}(z)}{\partial\nu} = \frac{1}{2\nu+1}[P_{\nu+1}(z) - P_{\nu-1}(z)] \tag{4.19}$$

and the following difference-differential relations:

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_{\nu+1}(z)}{\partial\nu} - z\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_\nu(z)}{\partial\nu} = (\nu+1)\frac{\partial P_\nu(z)}{\partial\nu} + P_\nu(z) \tag{4.20}$$

$$z\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_\nu(z)}{\partial\nu} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_{\nu-1}(z)}{\partial\nu} = \nu\frac{\partial P_\nu(z)}{\partial\nu} + P_\nu(z) \tag{4.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_{\nu+1}(z)}{\partial\nu} - \frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_{\nu-1}(z)}{\partial\nu} = (2\nu+1)\frac{\partial P_\nu(z)}{\partial\nu} + 2P_\nu(z) \tag{4.22}$$

$$(z+1)\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_\nu(z)}{\partial\nu} - (z+1)\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_{\nu-1}(z)}{\partial\nu} = \nu\frac{\partial P_\nu(z)}{\partial\nu} + \nu\frac{\partial P_{\nu-1}(z)}{\partial\nu} + P_\nu(z) + P_{\nu-1}(z) \tag{4.23}$$

$$(z^2-1)\frac{\mathrm{d}}{\mathrm{d}z}\frac{\partial P_\nu(z)}{\partial\nu} = \nu z\frac{\partial P_\nu(z)}{\partial\nu} - \nu\frac{\partial P_{\nu-1}(z)}{\partial\nu} + zP_\nu(z) - P_{\nu-1}(z). \tag{4.24}$$

We shall refer to the relations (4.15)–(4.24) in later sections.

5. Formulae for $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ and some related problems

We have already mentioned in the introduction that analysing mathematically some physical problems one encounters the derivative $\partial P_\nu(z)/\partial\nu$ evaluated at integer values of ν, i.e., $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ or $[\partial P_\nu(z)/\partial\nu]_{\nu=-n-1}$. Below, we shall study this particular case. Since, in virtue of the property (4.16), it holds that

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=-n-1} = -\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n} \tag{5.1}$$

we shall be concerned with $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ only.

5.1. The Jolliffe's formula (2.3)

We begin with the observation that, because of equation (4.10), the derivative $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ may be rewritten as

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n} = \frac{1}{2^n\pi\mathrm{i}}\oint_{\mathcal{C}^{(+)}}\mathrm{d}t\,\frac{(t^2-1)^n}{(t-z)^{n+1}}\ln\frac{t+1}{2} - P_n(z)\ln\frac{z+1}{2}. \tag{5.2}$$

In this equation, the only singularity of the integrand within the region enclosed by the contour $\mathcal{C}^{(+)}$ is the pole at $t = z$ of order $n + 1$. Hence, the integral may be immediately evaluated by the method of residues. This gives

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n} = \frac{1}{2^{n-1}n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z^2-1)^n\ln\frac{z+1}{2}\right] - P_n(z)\ln\frac{z+1}{2} \tag{5.3}$$

which is the Jolliffe's formula (2.3). It is worthwhile to mention that the validity of this formula was proved by Jolliffe in [25] in an entirely different way, without resorting to the complex variable techniques.

5.2. Polynomials $R_n(z)$

5.2.1. General considerations. For reasons which will become clear shortly, it is convenient to write $[\partial P_\nu(z)/\partial\nu]_{\nu=n}$ in the form

$$\left.\frac{\partial P_\nu(z)}{\partial\nu}\right|_{\nu=n} = P_n(z)\ln\frac{z+1}{2} + R_n(z). \tag{5.4}$$

According to equations (4.9) and (3.1), the function $R_n(z)$ may be represented as the contour integral

$$R_n(z) = \frac{1}{2^n\pi\mathrm{i}}\oint_{\mathcal{C}^{(+)}}\mathrm{d}t\,\frac{(t^2-1)^n}{(t-z)^{n+1}}\ln\frac{t+1}{z+1}. \tag{5.5}$$

Since the only singularity of the integrand within the domain enclosed by $\mathcal{C}^{(+)}$ is the pole at $t = z$ of order $n + 1$, on applying the residue theorem we have

$$R_n(z) = \frac{1}{2^{n-1}n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left[(t^2-1)^n\ln\frac{t+1}{z+1}\right]_{t=z}. \tag{5.6}$$

Carrying out the n-fold differentiation with the aid of the Leibniz identity transforms equation (5.6) into

$$R_n(z) = \frac{1}{2^{n-1}n!}\sum_{k=1}^{n}(-1)^{k+1}(k-1)!\binom{n}{k}\frac{1}{(z+1)^k}\frac{\mathrm{d}^{n-k}(z^2-1)^n}{\mathrm{d}z^{n-k}}. \tag{5.7}$$

(Observe that we might arrive at the above result manipulating with the Jolliffe's formula (5.3) with the aid of the Leibniz identity and the Rodrigues formula (3.5), rather than evaluating the contour integral (5.5).) In equation (5.7), and hereafter, we adopt the convention that if the upper limit of the sum is less by unity than the lower one, then the sum vanishes identically. The derivative in the summand appears to be identical with that met in the Rodrigues formula for the Gegenbauer polynomial

$$C_{n-k}^{(k+1/2)}(z) = \frac{k!(n+k)!}{2^{n-k}n!(2k)!(n-k)!}\frac{1}{(z^2-1)^k}\frac{\mathrm{d}^{n-k}(z^2-1)^n}{\mathrm{d}z^{n-k}} = \frac{2^k k!}{(2k)!}\frac{\mathrm{d}^k P_n(z)}{\mathrm{d}z^k} \tag{5.8}$$

so that equation (5.7) may be rewritten as

$$R_n(z) = -n!\sum_{k=1}^{n}\frac{(2k)!}{2^{k-1}kk!(n+k)!}(1-z)^k C_{n-k}^{(k+1/2)}(z). \tag{5.9}$$

Table 1. The polynomials $R_n(z)$ with $0 \leqslant n \leqslant 6$.

n	$R_n(z)$
0	0
1	$z-1$
2	$\frac{7}{4}z^2 - \frac{3}{2}z - \frac{1}{4}$
3	$\frac{37}{12}z^3 - \frac{5}{2}z^2 - \frac{5}{4}z + \frac{2}{3}$
4	$\frac{533}{96}z^4 - \frac{35}{8}z^3 - \frac{59}{16}z^2 + \frac{55}{24}z + \frac{7}{32}$
5	$\frac{1627}{160}z^5 - \frac{63}{8}z^4 - \frac{449}{48}z^3 + \frac{49}{8}z^2 + \frac{47}{32}z - \frac{8}{15}$
6	$\frac{18107}{960}z^6 - \frac{231}{16}z^5 - \frac{1417}{64}z^4 + \frac{119}{8}z^3 + \frac{379}{64}z^2 - \frac{231}{80}z - \frac{37}{192}$

It is seen from equation (5.9) that the function $R_n(z)$ is a polynomial in z of degree n, possessing the property[2]

$$R_n(1) = 0. \tag{5.10}$$

The polynomials $R_n(z)$ with $0 \leqslant n \leqslant 6$ are shown explicitly in table 1.

In later sections, we shall derive several expressions for the polynomials $R_n(z)$, alternative to that in equation (5.9). However, before it is done, we have to know some functional properties of these polynomials. These are briefly discussed in the following section.

5.2.2. Some functional properties of the polynomials $R_n(z)$. If we particularize equation (4.17) to the case $\nu = n$ and substitute therein the representation (5.4), after exploiting the Legendre identity (3.8) we find

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1)\right] R_n(z) = 2(z-1)\frac{\mathrm{d}P_n(z)}{\mathrm{d}z} - 2nP_n(z). \tag{5.11}$$

With the aid of the recurrence relation (3.11), equation (5.11) may be transformed into

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1)\right] R_n(z) = 2\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z} - 2\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}. \tag{5.12}$$

However, it is known [5, equation (8.915.2)] that the Legendre polynomials obey

$$\frac{\mathrm{d}P_n(z)}{\mathrm{d}z} = \sum_{k=0}^{\mathrm{int}[(n-1)/2]} (2n-4k-1)P_{n-2k-1}(z). \tag{5.13}$$

Hence, it may be easily deduced that

$$\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z} - \frac{\mathrm{d}P_n(z)}{\mathrm{d}z} = \sum_{k=0}^{n-1}(-1)^{n+k}(2k+1)P_k(z) \tag{5.14}$$

[2] It seems worthwhile to add that if we introduce the functions $R_{-n-1}(z)$, defined through

$$\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=-n-1} = P_{-n-1}(z)\ln\frac{z+1}{2} + R_{-n-1}(z)$$

these are *not* polynomials in z. It follows from the above definition and from equations (5.1), (5.4) and (3.7) that

$$R_{-n-1}(z) = -2P_n(z)\ln\frac{z+1}{2} - R_n(z).$$

This observation will be of value in the context of equations (5.19)–(5.21).

and insertion of this result into equation (5.12) gives

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z}+n(n+1)\right]R_n(z)=2\sum_{k=0}^{n-1}(-1)^{n+k}(2k+1)P_k(z). \tag{5.15}$$

Similarly, if the representation (5.4) is plugged into the recurrence relations (4.18)–(4.24), one obtains the following three-term recurrence relation obeyed by the polynomials $R_n(z)$:

$$(n+1)R_{n+1}(z)-(2n+1)zR_n(z)+nR_{n-1}(z)=-P_{n+1}(z)+2zP_n(z)-P_{n-1}(z) \tag{5.16}$$

or equivalently

$$(n+1)R_{n+1}(z)-(2n+1)zR_n(z)+nR_{n-1}(z)=\frac{1}{2n+1}[P_{n+1}(z)-P_{n-1}(z)] \tag{5.17}$$

and further

$$(z+1)\frac{\mathrm{d}R_{n+1}(z)}{\mathrm{d}z}-z(z+1)\frac{\mathrm{d}R_n(z)}{\mathrm{d}z}=(n+1)(z+1)R_n(z)-P_{n+1}(z)+(2z+1)P_n(z) \tag{5.18}$$

$$z(z+1)\frac{\mathrm{d}R_n(z)}{\mathrm{d}z}-(z+1)\frac{\mathrm{d}R_{n-1}(z)}{\mathrm{d}z}=n(z+1)R_n(z)+P_n(z)+P_{n-1}(z) \tag{5.19}$$

$$\begin{aligned}(z+1)\frac{\mathrm{d}R_{n+1}(z)}{\mathrm{d}z}-(z+1)\frac{\mathrm{d}R_{n-1}(z)}{\mathrm{d}z}&=(2n+1)(z+1)R_n(z)\\&\quad-P_{n+1}(z)+2(z+1)P_n(z)+P_{n-1}(z)\end{aligned} \tag{5.20}$$

$$(z+1)\frac{\mathrm{d}R_n(z)}{\mathrm{d}z}-(z+1)\frac{\mathrm{d}R_{n-1}(z)}{\mathrm{d}z}=nR_n(z)+nR_{n-1}(z)+2P_{n-1}(z) \tag{5.21}$$

$$(z^2-1)\frac{\mathrm{d}R_n(z)}{\mathrm{d}z}=nzR_n(z)-nR_{n-1}(z)+P_n(z)-P_{n-1}(z). \tag{5.22}$$

If $n=0$, in equations (5.19)–(5.21) one should set

$$\frac{\mathrm{d}R_{-1}(z)}{\mathrm{d}z}=-\frac{2}{z+1} \tag{5.23}$$

in accordance with what has been said in footnote 2.

5.2.3. A generating function for $R_n(z)$. In this section, we shall attempt to sum to the closed form the series

$$F(h;z)=\sum_{n=0}^{\infty}h^nR_n(z). \tag{5.24}$$

If the domain of convergence of the series includes some neighbourhood of the point $h=0$, the function $F(h;z)$ may be used to generate the polynomials $R_n(z)$, since then it holds that

$$R_n(z)=\frac{1}{n!}\frac{\partial^n F(h;z)}{\partial h^n}\bigg|_{h=0}. \tag{5.25}$$

To find the function $F(h;z)$, we shall assume temporarily that $|h|\ll 1$ (later, once the function $F(h;z)$ is found, we shall establish the actual radius of convergence of the expansion (5.24) by referring to facts known from the complex analysis). We begin with the observation that the value of the integral on the right-hand side of equation (5.5) is not changed if the contour $\mathcal{C}^{(+)}$ is deformed into the small circumference

$$|t-z|=|h(z^2-1)| \tag{5.26}$$

oriented in the positive sense and denoted hereafter as $\mathcal{C}_h^{(+)}$. Substituting $R_n(z)$ in the form (5.5), but with $\mathcal{C}^{(+)}$ replaced by $\mathcal{C}_h^{(+)}$, into the expansion (5.24), and interchanging subsequently the orders of summation and integration, yields

$$F(h;z)=\frac{1}{\pi \mathrm{i}}\oint_{\mathcal{C}_h^{(+)}}\mathrm{d}t\left[\sum_{n=0}^{\infty}\left(\frac{h}{2}\frac{t^2-1}{t-z}\right)^n\right]\frac{1}{t-z}\ln\frac{t+1}{z+1}. \tag{5.27}$$

On the contour (5.26) it holds that

$$\left|\frac{h}{2}\frac{t^2-1}{t-z}\right|=\frac{1}{2}+O(h)<1 \tag{5.28}$$

so that the series under the integral converges and may be summed to the closed form

$$\sum_{n=0}^{\infty}\left(\frac{h}{2}\frac{t^2-1}{t-z}\right)^n=\frac{1}{1-\dfrac{h}{2}\dfrac{t^2-1}{t-z}}=-2\frac{t-z}{h(t-t_+)(t-t_-)} \tag{5.29}$$

with

$$t_\pm=\frac{1\pm\sqrt{h^2-2hz+1}}{h}=\frac{2z-h}{1\mp\sqrt{h^2-2hz+1}} \tag{5.30}$$

where we choose this branch of the square root for which

$$\lim_{h\to 0}\sqrt{h^2-2hz+1}=+1. \tag{5.31}$$

Hence, after combining equations (5.27) and (5.29), we arrive at

$$F(h;z)=\frac{2\mathrm{i}}{\pi h}\oint_{\mathcal{C}_h^{(+)}}\mathrm{d}t\,\frac{1}{(t-t_+)(t-t_-)}\ln\frac{t+1}{z+1}. \tag{5.32}$$

The integrand in equation (5.32) has two simple poles at $t=t_\pm$ (and two branch points located at $t=-1$ and $|t|=\infty$). Since it follows from equation (5.30) that

$$t_+=2h^{-1}-z+O(h)\qquad(|h|\ll 1) \tag{5.33}$$

and

$$t_-=z+\tfrac{1}{2}h(z^2-1)+O(h^2)\qquad(|h|\ll 1) \tag{5.34}$$

it appears that only t_- lies in the domain enclosed by the contour (5.26). Hence, evaluating the integral by residues, we obtain

$$F(h;z)=\frac{4}{h}\frac{1}{t_+-t_-}\ln\frac{t_-+1}{z+1} \tag{5.35}$$

or equivalently, after exploiting equation (5.30),

$$F(h;z)=2\frac{1}{\sqrt{h^2-2hz+1}}\ln\frac{h+1-\sqrt{h^2-2hz+1}}{h(z+1)}. \tag{5.36}$$

It remains to establish the radius of convergence of the expansion obtained by merging equations (5.24) and (5.36). The singularities of the function (5.36), considered as a function of the variable h, are two branch points associated with the square root and located at $z\pm\sqrt{z^2-1}$. Hence, on the ground of the well-known theorem [30, section 5.4] concerning the Taylor expansion in the complex domain, we state that the expansion converges in the circle

$$|h|<\min\left|z\pm\sqrt{z^2-1}\right|. \tag{5.37}$$

Thus, in summary, we have derived the formula

$$2\frac{1}{\sqrt{h^2-2hz+1}}\ln\frac{h+1-\sqrt{h^2-2hz+1}}{h(z+1)}=\sum_{n=0}^{\infty}h^nR_n(z)\qquad\left(|h|<\min\left|z\pm\sqrt{z^2-1}\right|\right).\tag{5.38}$$

If we replace in equation (5.38) h by h^{-1}, in virtue of the fact that

$$\frac{1}{z\pm\sqrt{z^2-1}}=z\mp\sqrt{z^2-1}\tag{5.39}$$

we infer the expansion

$$2\frac{1}{\sqrt{h^2-2hz+1}}\ln\frac{h+1-\sqrt{h^2-2hz+1}}{z+1}=\sum_{n=0}^{\infty}h^{-n-1}R_n(z)\qquad\left(|h|>\max\left|z\pm\sqrt{z^2-1}\right|\right)\tag{5.40}$$

which also may be deduced from the findings of Bromwich [18].

5.2.4. The Bromwich's formula (2.1). Let us define

$$G(h;z)=\ln\frac{h+1-\sqrt{h^2-2hz+1}}{h(z+1)}.\tag{5.41}$$

To find the power series representation of $G(h;z)$ with respect to h, we observe that differentiating the defining equation (5.41) with respect to z gives

$$\frac{\partial G(h;z)}{\partial z}=\frac{1}{2(z+1)}\left(\frac{h+1}{\sqrt{h^2-2hz+1}}-1\right).\tag{5.42}$$

With the aid of the Legendre expansion

$$\frac{1}{\sqrt{h^2-2hz+1}}=\sum_{n=0}^{\infty}h^nP_n(z)\qquad\left(|h|<\min\left|z\pm\sqrt{z^2-1}\right|\right),\tag{5.43}$$

equation (5.42) may be rewritten as

$$\frac{\partial G(h;z)}{\partial z}=\frac{1}{2(z+1)}\sum_{n=1}^{\infty}h^n[P_n(z)+P_{n-1}(z)]\qquad\left(|h|<\min\left|z\pm\sqrt{z^2-1}\right|\right).\tag{5.44}$$

However, from equation (3.13) it may be inferred that the Legendre polynomials obey

$$\frac{P_n(z)+P_{n-1}(z)}{z+1}=\frac{1}{n}\left[\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}-\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z}\right]\qquad(n\neq 0)\tag{5.45}$$

and consequently equation (5.44) may be cast into the form

$$\frac{\partial G(h;z)}{\partial z}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{h^n}{n}\left[\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}-\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z}\right]\qquad\left(|h|<\min\left|z\pm\sqrt{z^2-1}\right|\right).\tag{5.46}$$

Integration of equation (5.46) with respect to z gives

$$G(h;z)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{h^n}{n}[P_n(z)-P_{n-1}(z)]+g(h)\qquad\left(|h|<\min\left|z\pm\sqrt{z^2-1}\right|\right)\tag{5.47}$$

where $g(h)$ remains to be determined. Since the Legendre polynomials, being particular cases of the Legendre functions, possess the property (3.6), from equation (5.47) we infer that

$$g(h)=G(h;1).\tag{5.48}$$

If we couple this with equation (5.41) and recall that in our considerations we use this branch of the square root which obeys the limiting relation (5.31), we find

$$g(h) = \ln \frac{h + 1 - (1 - h)}{2h} \equiv 0. \tag{5.49}$$

Hence, equations (5.41), (5.47) and (5.49) lead us to the expansion

$$\ln \frac{h + 1 - \sqrt{h^2 - 2hz + 1}}{h(z + 1)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{h^n}{n} [P_n(z) - P_{n-1}(z)] \qquad \left(|h| < \min \left|z \pm \sqrt{z^2 - 1}\right|\right). \tag{5.50}$$

If equations (5.36), (5.43) and (5.50) are combined, this yields the representation of the generating function $F(h; z)$ in the form of the product of two power series:

$$F(h; z) = \left[\sum_{n=0}^{\infty} h^n P_n(z)\right] \left\{\sum_{n=1}^{\infty} \frac{h^n}{n} [P_n(z) - P_{n-1}(z)]\right\} \qquad \left(|h| < \min \left|z \pm \sqrt{z^2 - 1}\right|\right). \tag{5.51}$$

Multiplying these two series gives

$$F(h; z) = \sum_{n=0}^{\infty} h^n \left\{\sum_{k=1}^{n} \frac{1}{k} [P_k(z) - P_{k-1}(z)] P_{n-k}(z)\right\} \qquad \left(|h| < \min \left|z \pm \sqrt{z^2 - 1}\right|\right) \tag{5.52}$$

and combining this result with equation (5.24) furnishes

$$R_n(z) = \sum_{k=1}^{n} \frac{1}{k} [P_k(z) - P_{k-1}(z)] P_{n-k}(z), \tag{5.53}$$

i.e., Bromwich's formula (2.1).

5.2.5. The Bromwich's formula (2.2). We have already proved that $R_n(z)$ is the polynomial in z of degree n. Hence, it must be possible to represent it as a linear combination of the Legendre polynomials in z, of degrees not exceeding n:

$$R_n(z) = \sum_{k=0}^{n} c_{nk} P_k(z). \tag{5.54}$$

To find the combination coefficients c_{nk}, we plug the expansion (5.54) into the differential identity (5.15) and simplify the result with the aid of the Legendre formula (3.8). Equating coefficients at $P_k(z)$ on both sides of the resulting relation furnishes the coefficients c_{nk}, with $0 \leqslant k \leqslant n - 1$, in the form

$$c_{nk} = (-1)^{n+k} \frac{2(2k + 1)}{(n - k)(n + k + 1)} \qquad (0 \leqslant k \leqslant n - 1). \tag{5.55}$$

To determine the coefficient c_{nn}, we exploit the fact that the polynomials $R_n(z)$ have been shown (cf equation (5.10)) to vanish at $z = 1$. In virtue of the property (3.6), we thus have the sum rule

$$\sum_{k=0}^{n} c_{nk} = 0 \tag{5.56}$$

from which, after utilizing the result (5.55), it follows that

$$c_{nn} = 2\sum_{k=0}^{n-1}(-1)^{n+k+1}\frac{2k+1}{(n-k)(n+k+1)}. \tag{5.57}$$

Equations (5.54), (5.55) and (5.57) lead to the following representation of the polynomial $R_n(z)$:

$$R_n(z) = 2\sum_{k=0}^{n-1}(-1)^{n+k}\frac{2k+1}{(n-k)(n+k+1)}[P_k(z) - P_n(z)] \tag{5.58}$$

which is Bromwich's formula (2.2).

Before concluding this section, we remark that with no difficulty it may be proved that the coefficient c_{nn} may be expressed not only as in equation (5.57), but also in several other equivalent forms, for instance as

$$c_{nn} = \sum_{k=0}^{n-1}\frac{1}{(k+1)(2k+1)} \tag{5.59}$$

$$c_{nn} = 2\sum_{k=1}^{2n}(-1)^{k+1}\frac{1}{k} \tag{5.60}$$

or

$$c_{nn} = 2\sum_{k=n+1}^{2n}\frac{1}{k}. \tag{5.61}$$

From the last of the above equations and from the following well-known property of the digamma function [4]

$$\psi(n+1) = -\gamma + \sum_{k=1}^{n}\frac{1}{k} \tag{5.62}$$

where γ is the Euler–Mascheroni constant, it follows that

$$c_{nn} = 2[\psi(2n+1) - \psi(n+1)] \tag{5.63}$$

so that equation (5.58) may be also rewritten as

$$R_n(z) = 2[\psi(2n+1) - \psi(n+1)]P_n(z) + 2\sum_{k=0}^{n-1}(-1)^{n+k}\frac{2k+1}{(n-k)(n+k+1)}P_k(z). \tag{5.64}$$

We shall make use of equation (5.64) in sections 5.3.1 and 6.2.

5.2.6. The Schelkunoff's formula (2.4). Next, let us seek the polynomial $R_n(z)$ in the form

$$R_n(z) = \sum_{k=0}^{n} a_{nk}^{(-)}(z-1)^k. \tag{5.65}$$

By virtue of the property (5.10) it is immediately seen that it must hold that

$$a_{n0}^{(-)} = 0. \tag{5.66}$$

To find the coefficients $a_{nk}^{(-)}$ with $1 \leqslant k \leqslant n$, we shall exploit the differential identity (5.12) obeyed by $R_n(z)$. If use is made of the first Murphy's formula [11, section 15]

$$P_n(z) = \sum_{k=0}^{n} \frac{(n+k)!}{(k!)^2(n-k)!} \left(\frac{z-1}{2}\right)^k, \tag{5.67}$$

one finds that

$$\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z} - \frac{\mathrm{d}P_n(z)}{\mathrm{d}z} = -\sum_{k=0}^{n-1} \frac{(n+k)!}{(k!)^2(n-k-1)!} \left(\frac{z-1}{2}\right)^k \tag{5.68}$$

so that equation (5.12) becomes

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1)\right] R_n(z) = -\sum_{k=0}^{n-1} \frac{(n+k)!}{2^{k-1}(k!)^2(n-k-1)!}(z-1)^k. \tag{5.69}$$

Substituting here the expansion (5.65) and equating coefficients at $(z-1)^k$ on both sides of the resulting equation, one arrives at the inhomogeneous two-term difference relation

$$2(k+1)^2 a_{n,k+1}^{(-)} - (n-k)(n+k+1)a_{nk}^{(-)} = \frac{(n+k)!}{2^{k-1}(k!)^2(n-k-1)!} \qquad (0 \leqslant k \leqslant n-1). \tag{5.70}$$

The structure of the inhomogeneity suggests to seek $a_{nk}^{(-)}$ in the form

$$a_{nk}^{(-)} = \frac{(n+k)!}{2^{k-1}(k!)^2(n-k)!} b_{nk}^{(-)} \tag{5.71}$$

with $b_{nk}^{(-)}$ to be determined. Plugging equation (5.71) into (5.70) yields the following simple recurrence relation for the coefficients $b_{nk}^{(-)}$:

$$b_{n,k+1}^{(-)} - b_{nk}^{(-)} = \frac{1}{n+k+1} \qquad (0 \leqslant k \leqslant n-1) \tag{5.72}$$

which is to be solved subject to the initial condition

$$b_{n0}^{(-)} = 0 \tag{5.73}$$

implied by equations (5.71) and (5.66). The solution is straightforwardly found to be

$$b_{nk}^{(-)} = \sum_{m=1}^{k} \frac{1}{n+m} = \psi(n+k+1) - \psi(n+1). \tag{5.74}$$

On combining equations (5.65), (5.71) and (5.74), one arrives at the Schelkunoff's formula

$$R_n(z) = 2\sum_{k=1}^{n} \frac{(n+k)!}{(k!)^2(n-k)!}[\psi(n+k+1) - \psi(n+1)] \left(\frac{z-1}{2}\right)^k \tag{5.75}$$

(cf equation (2.4) and the footnote thereto).

5.2.7. An analogue of the Schelkunoff's formula (2.4). Having found in the preceding section the expansion of $R_n(z)$ in powers of $z-1$, it is then natural to consider an analogous expansion in powers of $z+1$:

$$R_n(z) = \sum_{k=0}^{n} a_{nk}^{(+)}(z+1)^k. \tag{5.76}$$

To find the coefficients $a_{nk}^{(+)}$, we shall proceed as in the preceding section, but this time starting with the second of Murphy's formulae [11, section 15]:

$$P_n(z) = \sum_{k=0}^{n} (-1)^{n+k} \frac{(n+k)!}{(k!)^2(n-k)!} \left(\frac{z+1}{2}\right)^k \tag{5.77}$$

from which it follows that

$$\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z} - \frac{\mathrm{d}P_n(z)}{\mathrm{d}z} = \sum_{k=0}^{n-1} (-1)^{n+k} \frac{n(n+k)!}{k!(k+1)!(n-k-1)!} \left(\frac{z+1}{2}\right)^k. \tag{5.78}$$

Combining this result with equation (5.12) casts the latter into the form

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1)\right] R_n(z) = \sum_{k=0}^{n-1} (-1)^{n+k} \frac{n(n+k)!}{2^{k-1}k!(k+1)!(n-k-1)!}(z+1)^k. \tag{5.79}$$

Plugging here the expansion (5.76) and equating then coefficients at $(z+1)^k$ on both sides of the resulting equation leads to the two-term recurrence

$$2(k+1)^2 a_{n,k+1}^{(+)} + (n-k)(n+k+1)a_{nk}^{(+)} = (-1)^{n+k} \frac{n(n+k)!}{2^{k-1}k!(k+1)!(n-k-1)!} \qquad (0 \leqslant k \leqslant n-1). \tag{5.80}$$

Guided by the form of the inhomogeneity, we substitute

$$a_{nk}^{(+)} = (-1)^{n+k+1} \frac{(n+k)!}{2^{k-1}(k!)^2(n-k)!} b_{nk}^{(+)} \tag{5.81}$$

which results in the much simpler recurrence

$$b_{n,k+1}^{(+)} - b_{nk}^{(+)} = \frac{1}{k+1} - \frac{1}{n+k+1} \qquad (0 \leqslant k \leqslant n-1). \tag{5.82}$$

The solution to equation (5.82), expressed in terms of the still unknown coefficient $b_{n0}^{(+)}$, is easily found to be

$$b_{nk}^{(+)} = b_{n0}^{(+)} + \psi(k+1) - \psi(1) + \psi(n+1) - \psi(n+k+1). \tag{5.83}$$

If, for brevity, we define

$$\bar{b}_{n0}^{(+)} = b_{n0}^{(+)} - \psi(1) + \psi(n+1) \tag{5.84}$$

equation (5.83) becomes

$$b_{nk}^{(+)} = \bar{b}_{n0}^{(+)} + \psi(k+1) - \psi(n+k+1). \tag{5.85}$$

To determine $\bar{b}_{n0}^{(+)}$, and subsequently $b_{nk}^{(+)}$ and $a_{nk}^{(+)}$, first we observe that it follows from equations (5.54), (5.76) and (5.77) that the coefficients $a_{nn}^{(+)}$ and c_{nn} are related through

$$a_{nn}^{(+)} = \frac{(2n)!}{2^n(n!)^2} c_{nn}. \tag{5.86}$$

Particularizing then equations (5.81) and (5.85) to the case $k = n$ and merging with equation (5.86) results in the relationship

$$-2\left[\bar{b}_{n0}^{(+)} + \psi(n+1) - \psi(2n+1)\right] = c_{nn}. \tag{5.87}$$

Solving this for $\bar{b}_{n0}^{(+)}$ and using c_{nn} in the form (5.63) gives

$$\bar{b}_{n0}^{(+)} = 0. \tag{5.88}$$

Inserting this into equation (5.85) we have

$$b_{nk}^{(+)} = \psi(k+1) - \psi(n+k+1) \tag{5.89}$$

and combining this result with equations (5.81) and (5.76) we eventually arrive at the sought expansion of $R_n(z)$ in powers of $z+1$:

$$R_n(z) = 2\sum_{k=0}^{n}(-1)^{n+k}\frac{(n+k)!}{(k!)^2(n-k)!}[\psi(n+k+1) - \psi(k+1)]\left(\frac{z+1}{2}\right)^k. \tag{5.90}$$

5.3. Polynomials $V_n(z)$ and $W_{n-1}(z)$

In this section, we shall make a brief study of the properties of the polynomials $V_n(z)$ and $W_{n-1}(z)$, defined in terms of $R_n(z)$ as

$$V_n(z) = \tfrac{1}{2}[R_n(z) + (-1)^n R_n(-z)] \tag{5.91}$$

and[3]

$$W_{n-1}(z) = -\tfrac{1}{2}[R_n(z) - (-1)^n R_n(-z)] \tag{5.92}$$

respectively, so that it holds that

$$R_n(z) = V_n(z) - W_{n-1}(z). \tag{5.93}$$

It is evident from the definitions (5.91) and (5.92) that degrees of the polynomials $V_n(z)$ and $W_{n-1}(z)$ are equal to n and $n-1$, respectively.

While we are not aware of any investigation on the polynomials $V_n(z)$, we shall show below that the polynomials $W_{n-1}(z)$ are essentially the Christoffel polynomials, well known from the theory of the Legendre functions of the second kind [11, section 34].

5.3.1. Some functional properties of the polynomials $V_n(z)$. It is an immediate consequence of the definition (5.91) that the polynomials $V_n(z)$ possess the property

$$V_n(-z) = (-1)^n V_n(z), \tag{5.94}$$

i.e., the parity of the polynomial $V_n(z)$ is the same as that of the Legendre polynomial $P_n(z)$. Further, combining equation (5.91) with the Bromwich formulae (2.1) and (2.2), with the Schelkunoff formula (2.4) and with the analogue (5.90) of the latter, we find the following respective representations of the polynomials $V_n(z)$:

$$V_n(z) = \sum_{k=1}^{n}\frac{1}{k}P_k(z)P_{n-k}(z) \tag{5.95}$$

$$V_n(z) = 2[\psi(2n+1) - \psi(n+1)]P_n(z) + 2\sum_{k=0}^{n-2}\frac{1+(-1)^{n+k}}{2}\frac{2k+1}{(n-k)(n+k+1)}P_k(z) \tag{5.96}$$

or, equivalently,

$$V_n(z) = 2[\psi(2n+1) - \psi(n+1)]P_n(z) + \sum_{k=0}^{\mathrm{int}[(n-2)/2]}\frac{2n-4k-3}{(k+1)(2n-2k-1)}P_{n-2k-2}(z) \tag{5.97}$$

[3] The polynomials $W_n(z)$ from [23, 24] are identical with the polynomials $R_n(z)$ from the present paper and should not be confused with the Christoffel polynomials defined in equation (5.92).

Table 2. The polynomials $V_n(z)$ with $0 \leqslant n \leqslant 6$.

n	$V_n(z)$
0	0
1	z
2	$\frac{7}{4}z^2 - \frac{1}{4}$
3	$\frac{37}{12}z^3 - \frac{5}{4}z$
4	$\frac{533}{96}z^4 - \frac{59}{16}z^2 + \frac{7}{32}$
5	$\frac{1627}{160}z^5 - \frac{449}{48}z^3 + \frac{47}{32}z$
6	$\frac{18107}{960}z^6 - \frac{1417}{64}z^4 + \frac{379}{64}z^2 - \frac{37}{192}$

$$V_n(z) = \sum_{k=0}^{n} \frac{(n+k)!}{(k!)^2(n-k)!}[2\psi(n+k+1) - \psi(n+1) - \psi(k+1)]\left(\frac{z-1}{2}\right)^k \tag{5.98}$$

and

$$V_n(z) = \sum_{k=0}^{n}(-1)^{n+k} \frac{(n+k)!}{(k!)^2(n-k)!}[2\psi(n+k+1) - \psi(n+1) - \psi(k+1)]\left(\frac{z+1}{2}\right)^k. \tag{5.99}$$

The polynomials $V_n(z)$ with $0 \leqslant n \leqslant 6$ are explicitly shown in table 2.

Further properties of the polynomials $V_n(z)$ follow from coupling equation (5.91) with formulae (5.12) and (5.16)–(5.22), occasionally with the help of some of the relations (3.9)–(3.14) and of equation (5.45). In this way, we obtain the differential identity

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1)\right]V_n(z) = 2\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z}, \tag{5.100}$$

the inhomogeneous three-term recurrence relation

$$(n+1)V_{n+1}(z) - (2n+1)zV_n(z) + nV_{n-1}(z) = -P_{n+1}(z) + 2zP_n(z) - P_{n-1}(z) + \delta_{n0}, \tag{5.101}$$

or, equivalently,

$$(n+1)V_{n+1}(z) - (2n+1)zV_n(z) + nV_{n-1}(z) = \frac{1}{2n+1}[P_{n+1}(z) - P_{n-1}(z)] + \delta_{n0}, \tag{5.102}$$

and the differential-difference relations

$$(n+1)\frac{\mathrm{d}V_{n+1}(z)}{\mathrm{d}z} - (n+1)z\frac{\mathrm{d}V_n(z)}{\mathrm{d}z} = (n+1)^2V_n(z) + P_n(z) - \frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z} \tag{5.103}$$

$$nz\frac{\mathrm{d}V_n(z)}{\mathrm{d}z} - n\frac{\mathrm{d}V_{n-1}(z)}{\mathrm{d}z} = n^2V_n(z) - \frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z} \tag{5.104}$$

$$n(n+1)\frac{\mathrm{d}V_{n+1}(z)}{\mathrm{d}z} - n(n+1)\frac{\mathrm{d}V_{n-1}(z)}{\mathrm{d}z} = n(n+1)(2n+1)V_n(z) + nP_n(z) - (2n+1)\frac{\mathrm{d}P_{n-1}(z)}{\mathrm{d}z} \tag{5.105}$$

$$(z^2-1)\frac{\mathrm{d}V_n(z)}{\mathrm{d}z} = nzV_n(z) - nV_{n-1}(z) - P_{n-1}(z) + \delta_{n0}. \tag{5.106}$$

The origin of the term δ_{n0} in equations (5.101), (5.102) and (5.106), and also in some other formulae in this and in the following section, is due to the fact that $P_{-1}(z) \equiv P_0(z)$ is an *even* function of z.

Still another useful relationship, which will find an application in section 6.3, may be derived from the recurrences (5.101) and (3.9), with the latter specialized to the case $\nu = n$. Multiplying the former recurrence by $P_n(z)$ and the latter by $V_n(z)$, subtracting and exploiting the fact that

$$P_n(z)\delta_{n0} = \delta_{n0} \tag{5.107}$$

we find

$$\begin{aligned}(n+1)[P_n(z)V_{n+1}(z) - V_n(z)P_{n+1}(z)] - n[P_{n-1}(z)V_n(z) - V_{n-1}(z)P_n(z)]\\ = -P_n(z)P_{n+1}(z) + 2z[P_n(z)]^2 - P_{n-1}(z)P_n(z) + \delta_{n0}.\end{aligned} \tag{5.108}$$

Replacing in equation (5.108) n with k and summing over k from $k = 0$ to $k = n$ yields

$$(n+1)[P_n(z)V_{n+1}(z) - V_n(z)P_{n+1}(z)] = P_n(z)P_{n+1}(z) + 2\sum_{k=0}^{n} P_k(z)[zP_k(z) - P_{k+1}(z)]. \tag{5.109}$$

Since it holds that

$$zP_0(z) = P_1(z) \tag{5.110}$$

the term with $k = 0$ in the sum on the right-hand side of equation (5.109) may be dropped and this results in the sought relationship

$$(n+1)[P_n(z)V_{n+1}(z) - V_n(z)P_{n+1}(z)] = P_n(z)P_{n+1}(z) + 2\sum_{k=1}^{n} P_k(z)[zP_k(z) - P_{k+1}(z)]. \tag{5.111}$$

We conclude this section observing that from equations (5.91), (5.38) and (5.40) the following expansions may be inferred:

$$-\frac{1}{\sqrt{h^2-2hz+1}} \ln \frac{1-hz+\sqrt{h^2-2hz+1}}{2} = \sum_{n=0}^{\infty} h^n V_n(z) \qquad \left(|h| < \min\left|z \pm \sqrt{z^2-1}\right|\right) \tag{5.112}$$

$$-\frac{1}{\sqrt{h^2-2hz+1}} \ln \frac{h-z+\sqrt{h^2-2hz+1}}{2h} = \sum_{n=0}^{\infty} h^{-n-1} V_n(z) \qquad \left(|h| > \max\left|z \pm \sqrt{z^2-1}\right|\right). \tag{5.113}$$

5.3.2. Some functional properties of the polynomials $W_{n-1}(z)$. It is seen from equations (5.92) and (5.91) that the parity of the polynomial $W_{n-1}(z)$ is opposite to that of the polynomial $V_n(z)$ (hence, also to that of the Legendre polynomial $P_n(z)$):

$$W_{n-1}(-z) = (-1)^{n+1} W_{n-1}(z). \tag{5.114}$$

Table 3. The polynomials $W_{n-1}(z)$ with $-1 \leqslant n-1 \leqslant 5$.

$n-1$	$W_{n-1}(z)$
-1	0
0	1
1	$\frac{3}{2}z$
2	$\frac{5}{2}z^2 - \frac{2}{3}$
3	$\frac{35}{8}z^3 - \frac{55}{24}z$
4	$\frac{63}{8}z^4 - \frac{49}{8}z^2 + \frac{8}{15}$
5	$\frac{231}{16}z^5 - \frac{119}{8}z^3 + \frac{231}{80}z$

Equations (5.92), (2.1), (2.2), (2.4) and (5.90) imply the following representations of the polynomials $W_{n-1}(z)$:

$$W_{n-1}(z) = \sum_{k=1}^{n} \frac{1}{k} P_{k-1}(z) P_{n-k}(z) \tag{5.115}$$

$$W_{n-1}(z) = 2\sum_{k=0}^{n-1} \frac{1-(-1)^{n+k}}{2} \frac{2k+1}{(n-k)(n+k+1)} P_k(z) \tag{5.116}$$

$$W_{n-1}(z) = \sum_{k=0}^{\mathrm{int}[(n-1)/2]} \frac{2n-4k-1}{(2k+1)(n-k)} P_{n-2k-1}(z) \tag{5.117}$$

$$W_{n-1}(z) = \sum_{k=0}^{n-1} \frac{(n+k)!}{(k!)^2(n-k)!} [\psi(n+1) - \psi(k+1)] \left(\frac{z-1}{2}\right)^k \tag{5.118}$$

and

$$W_{n-1}(z) = \sum_{k=0}^{n-1} (-1)^{n+k+1} \frac{(n+k)!}{(k!)^2(n-k)!} [\psi(n+1) - \psi(k+1)] \left(\frac{z+1}{2}\right)^k. \tag{5.119}$$

The polynomials $W_{n-1}(z)$ with $0 \leqslant n \leqslant 6$ are listed in table 3.

A glance at either of equations (5.115)–(5.119) reveals that $W_{n-1}(z)$ is just the Christoffel's polynomial encountered in the theory of the Legendre functions of the second kind (in this connection, cf section 6.1); the reason for introducing the minus sign on the right-hand side of the definition (5.92) is now clear. We feel obliged to mention that the representation of $W_{n-1}(z)$ given in equation (5.117) is the one discovered by Christoffel, that in equation (5.115) is due to Hermite [31], while the one in equation (5.118) was found by Stieltjes [32, 33].

The following differential

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}}{\mathrm{d}z} + n(n+1)\right] W_{n-1}(z) = 2\frac{\mathrm{d}P_n(z)}{\mathrm{d}z} \tag{5.120}$$

recurrence

$$(n+1)W_n(z) - (2n+1)zW_{n-1}(z) + nW_{n-2}(z) = \delta_{n0} \tag{5.121}$$

and differential-difference

$$(n+1)\frac{\mathrm{d}W_n(z)}{\mathrm{d}z} - (n+1)z\frac{\mathrm{d}W_{n-1}(z)}{\mathrm{d}z} = (n+1)^2 W_{n-1}(z) - \frac{\mathrm{d}P_n(z)}{\mathrm{d}z} \tag{5.122}$$

$$nz\frac{\mathrm{d}W_{n-1}(z)}{\mathrm{d}z} - n\frac{\mathrm{d}W_{n-2}(z)}{\mathrm{d}z} = n^2 W_{n-1}(z) - \frac{\mathrm{d}P_n(z)}{\mathrm{d}z} \tag{5.123}$$

$$n(n+1)\frac{\mathrm{d}W_n(z)}{\mathrm{d}z} - n(n+1)\frac{\mathrm{d}W_{n-2}(z)}{\mathrm{d}z} = n(n+1)(2n+1)W_{n-1}(z) - (2n+1)\frac{\mathrm{d}P_n(z)}{\mathrm{d}z} \tag{5.124}$$

$$(z^2-1)\frac{\mathrm{d}W_{n-1}(z)}{\mathrm{d}z} = nzW_{n-1}(z) - nW_{n-2}(z) - P_n(z) + \delta_{n0} \tag{5.125}$$

relations obeyed by the Christoffel polynomials are readily deduced from the definition (5.92), the properties (5.12) and (5.16)–(5.22) and also, when needed, the relations (3.9)–(3.14) and (5.45).

A further relationship, of use in section 6.3, obeyed by the Christoffel polynomials is

$$(1-z^2)\left[P_n(z)\frac{\mathrm{d}W_{n-1}(z)}{\mathrm{d}z} - W_{n-1}(z)\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}\right] = [P_n(z)]^2 - 1. \tag{5.126}$$

To prove it, one multiplies equation (3.8), particularized to the case $\nu = n$, by $W_{n-1}(z)$, equation (5.120) by $P_n(z)$, subtracts and integrates with respect to z, obtaining

$$(1-z^2)\left[P_n(z)\frac{\mathrm{d}W_{n-1}(z)}{\mathrm{d}z} - W_{n-1}(z)\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}\right] = [P_n(z)]^2 + C \tag{5.127}$$

where C is an integration constant remaining to be determined. If one sets $z = 1$ and uses the property (3.6), one finds

$$C = -1, \tag{5.128}$$

hence, equation (5.126) follows.

As in section 5.3.1, we conclude with the following series expansions involving the polynomials studied here:

$$\pm\frac{1}{\sqrt{h^2-2hz+1}}\ln\frac{z-h\mp\sqrt{h^2-2hz+1}}{z\mp 1} = \sum_{n=0}^{\infty} h^n W_{n-1}(z)$$
$$\left(|h| < \min\left|z \pm \sqrt{z^2-1}\right|\right) \tag{5.129}$$

$$\pm\frac{1}{\sqrt{h^2-2hz+1}}\ln\frac{hz-1\mp\sqrt{h^2-2hz+1}}{h(z\mp 1)} = \sum_{n=0}^{\infty} h^{-n-1} W_{n-1}(z)$$
$$\left(|h| > \max\left|z \pm \sqrt{z^2-1}\right|\right) \tag{5.130}$$

(the upper or lower sets of signs on the left-hand sides of equations (5.129) and (5.130) may be chosen at will). These expansions follow from combining the definition (5.92) with the expansions (5.38) and (5.40).

6. Some applications

6.1. Formulae for the Legendre function of the second kind

As[4] the first example of the utility of the results of section 5, we shall present here a non-standard derivation of two representations of the Legendre function of the second kind with a non-negative integer degree.

[4] Applications of some of the results of section 5 to constructing explicit expressions for generalized Green functions for the Helmholtz operator on the two-dimensional unit sphere and for the Legendre operator on the interval $[-1, 1]$ have been presented in [24].

We begin with recalling that for $\nu \in \mathbb{C}\backslash\mathbb{Z}$ and $z \in \mathbb{C}\backslash\mathbb{R}$ the Legendre function of the second kind, $Q_\nu(z)$, may be defined in terms of the functions $P_\nu(\pm z)$ as [1, chapter 3]

$$Q_\nu(z) = \frac{\pi}{2}\frac{\mathrm{e}^{\mp\mathrm{i}\pi\nu}P_\nu(z) - P_\nu(-z)}{\sin(\pi\nu)} \tag{6.1}$$

with the upper (lower) sign being taken for $\mathrm{Im}(z) > 0$ (respectively $\mathrm{Im}(z) < 0$). As it is, the formula in equation (6.1) is not directly applicable to the case when $\nu = n \in \mathbb{N}$. However, on exploiting the complex version of the l'Hôpital rule, we infer that

$$Q_n(z) = \frac{1}{2}\frac{\partial P_\nu(z)}{\partial\nu}\bigg|_{\nu=n} - \frac{(-1)^n}{2}\frac{\partial P_\nu(-z)}{\partial\nu}\bigg|_{\nu=n} \mp \frac{1}{2}\mathrm{i}\pi P_n(z). \tag{6.2}$$

If in this equation use is made, twice, of the Jolliffe's formula (5.3), with the aid of the property

$$P_n(-z) = (-1)^n P_n(z), \tag{6.3}$$

one obtains

$$Q_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z^2-1)^n \ln\frac{1+z}{1-z}\right] - \frac{1}{2}P_n(z)\ln\frac{1+z}{1-z} \mp \frac{1}{2}\mathrm{i}\pi P_n(z). \tag{6.4}$$

Since it holds that

$$1 - z = (z-1)\,\mathrm{e}^{\mp\mathrm{i}\pi} \tag{6.5}$$

equation (6.4) may be simplified, yielding

$$Q_n(z) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[(z^2-1)^n \ln\frac{z+1}{z-1}\right] - \frac{1}{2}P_n(z)\ln\frac{z+1}{z-1} \tag{6.6}$$

(cf [11, section 44]). If, instead of combining equation (6.2) with the Jolliffe's formula (5.3), we shall use equation (5.4) and the definition (5.92), again with the aid of the property (6.3) we find

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\frac{1+z}{1-z} - W_{n-1}(z) \mp \frac{1}{2}\mathrm{i}\pi P_n(z) \tag{6.7}$$

which, after exploiting equation (6.5), becomes

$$Q_n(z) = \frac{1}{2}P_n(z)\ln\frac{z+1}{z-1} - W_{n-1}(z). \tag{6.8}$$

This is the well-known Christoffel's formula for $Q_n(z)$ [1–5]. Equations (6.6) and (6.8) remain valid if z is continued to the half-line $(1, \infty)$.

On the interval $-1 \leqslant x \leqslant 1$, the Legendre function of the second kind may be defined as [1, chapter 3]

$$Q_\nu(x) = \frac{\pi}{2}\frac{P_\nu(x)\cos(\pi\nu) - P_\nu(-x)}{\sin(\pi\nu)}. \tag{6.9}$$

Proceeding analogously as above, we find

$$Q_n(x) = \frac{1}{2}\frac{\partial P_\nu(x)}{\partial\nu}\bigg|_{\nu=n} - \frac{(-1)^n}{2}\frac{\partial P_\nu(-x)}{\partial\nu}\bigg|_{\nu=n} \tag{6.10}$$

and then

$$Q_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n \ln\frac{1+x}{1-x}\right] - \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} \tag{6.11}$$

$$Q_n(x) = \frac{1}{2}P_n(x)\ln\frac{1+x}{1-x} - W_{n-1}(x). \tag{6.12}$$

6.2. Summation of some Legendre series and evaluation of some definite integrals involving Legendre polynomials

Consider the Legendre series

$$S_n^{(\pm)}(x) = \sum_{\substack{k=0\\(k\neq n)}}^{\infty} (\pm 1)^k \frac{2k+1}{(n-k)(k+n+1)} P_k(x) \qquad (n \in \mathbb{N}). \tag{6.13}$$

We begin with the observation that the above equation may be rewritten as

$$S_n^{(\pm)}(x) = \lim_{\nu \to n} \frac{\partial}{\partial \nu} (\nu - n) \sum_{k=0}^{\infty} (\pm 1)^k \frac{2k+1}{(\nu-k)(k+\nu+1)} P_k(x) + \frac{(\pm 1)^n}{2n+1} P_n(x). \tag{6.14}$$

On the other hand, it is known [5, 34, 35] that

$$\sum_{k=0}^{\infty} (\pm 1)^k \frac{2k+1}{(\nu-k)(k+\nu+1)} P_k(x) = \frac{\pi}{\sin(\pi\nu)} P_\nu(\mp x) \qquad (\nu \notin \mathbb{Z}) \tag{6.15}$$

so that from equations (6.14) and (6.15) we have

$$S_n^{(\pm)}(x) = (-1)^n \left.\frac{\partial P_\nu(\mp x)}{\partial \nu}\right|_{\nu=n} + \frac{(\pm 1)^n}{2n+1} P_n(x). \tag{6.16}$$

Inserting here the results (5.4) and (5.64), after recalling the definition (6.13) and the property (6.3), one obtains the summation formula

$$\begin{aligned}\sum_{\substack{k=0\\(k\neq n)}}^{\infty} (\pm 1)^k \frac{2k+1}{(n-k)(k+n+1)} P_k(x) &= (\pm 1)^n P_n(x) \ln(1 \mp x) \\ &\quad + 2\sum_{k=0}^{n-1} (\pm 1)^k \frac{2k+1}{(n-k)(k+n+1)} P_k(x) \\ &\quad + (\pm 1)^n [\psi(2n+2) + \psi(2n+1) - 2\psi(n+1) - \ln 2] P_n(x).\end{aligned} \tag{6.17}$$

With no difficulty, it is possible to deduce from equation (6.17) two additional summation formulae:

$$\begin{aligned}\sum_{\substack{k=0\\(k\neq n)}}^{\infty} (\pm 1)^k \frac{2k+1}{|n-k|(k+n+1)} P_k(x) &= -(\pm 1)^n P_n(x) \ln(1 \mp x) \\ &\quad + (\pm 1)^n [\ln 2 + 2\psi(n+1) - \psi(2n+1) - \psi(2n+2)] P_n(x)\end{aligned} \tag{6.18}$$

and

$$\begin{aligned}\sum_{k=0}^{\infty} (\pm 1)^k \frac{2k+2n+3}{(k+1)(k+2n+2)} P_{k+n+1}(x) &= \mp P_n(x) \ln(1 \mp x) \\ &\quad - \sum_{k=0}^{n-1} (\pm 1)^{k+n+1} \frac{2k+1}{(n-k)(k+n+1)} P_k(x) \\ &\quad \pm [\ln 2 + 2\psi(n+1) - \psi(2n+1) - \psi(2n+2)] P_n(x).\end{aligned} \tag{6.19}$$

Further, if equation (6.18) is rewritten in the form

$$\begin{aligned}P_n(x) \ln(1 \mp x) &= -\sum_{\substack{k=0\\(k\neq n)}}^{\infty} (\pm 1)^{k+n} \frac{2k+1}{|n-k|(k+n+1)} P_k(x) \\ &\quad + [\ln 2 + 2\psi(n+1) - \psi(2n+1) - \psi(2n+2)] P_n(x)\end{aligned} \tag{6.20}$$

after multiplying the latter by the Legendre polynomial $P_{n'}(x)$, integrating the result over the interval $-1 \leqslant x \leqslant 1$ and utilizing the well-known orthogonality relation

$$\int_{-1}^{1} \mathrm{d}x\, P_n(x) P_{n'}(x) = \frac{2}{2n+1}\delta_{nn'} \tag{6.21}$$

we arrive at the definite integrals

$$\int_{-1}^{1} \mathrm{d}x\, P_n(x) P_{n'}(x) \ln(1 \mp x) = -(\pm 1)^{n+n'} \frac{2}{|n-n'|(n+n'+1)} \qquad (n \neq n') \tag{6.22}$$

and

$$\int_{-1}^{1} \mathrm{d}x\, [P_n(x)]^2 \ln(1 \mp x) = \frac{2}{2n+1}[\ln 2 + 2\psi(n+1) - \psi(2n+1) - \psi(2n+2)]. \tag{6.23}$$

6.3. Evaluation of the indefinite integral of the square of the Legendre polynomial

The last example we wish to consider is the evaluation of the indefinite integral of the Legendre polynomial squared. To achieve the goal, consider the Legendre identities

$$\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}P_\nu(z)}{\mathrm{d}z} + \nu(\nu+1)P_\nu(z) = 0 \tag{6.24}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}(1-z^2)\frac{\mathrm{d}P_{\nu'}(z)}{\mathrm{d}z} + \nu'(\nu'+1)P_{\nu'}(z) = 0. \tag{6.25}$$

Multiplying the former by $P_{\nu'}(z)$, the latter by $P_\nu(z)$, subtracting and integrating over z yields

$$\int \mathrm{d}z\, P_\nu(z) P_{\nu'}(z) = \frac{1-z^2}{(\nu-\nu')(\nu+\nu'+1)}\left[P_\nu(z)\frac{\mathrm{d}P_{\nu'}(z)}{\mathrm{d}z} - P_{\nu'}(z)\frac{\mathrm{d}P_\nu(z)}{\mathrm{d}z}\right] + C \tag{6.26}$$

where C is an arbitrary integration constant (henceforth, it will be tacitly assumed that C absorbs all additive constants). For $\nu' = \nu = n$, after utilizing the complex version of the l'Hospital rule, equation (6.26) becomes

$$\int \mathrm{d}z\, [P_n(z)]^2 = \frac{1}{2n+1}(1-z^2)\left[\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n} \frac{\mathrm{d}P_n(z)}{\mathrm{d}z} - P_n(z)\frac{\mathrm{d}}{\mathrm{d}z}\left.\frac{\partial P_\nu(z)}{\partial \nu}\right|_{\nu=n}\right] + C. \tag{6.27}$$

Transforming the right-hand side of this result with the aid of equation (5.4) gives the sought integral in the form

$$\begin{aligned}\int \mathrm{d}z\, [P_n(z)]^2 &= \frac{1}{2n+1}(z-1)[P_n(z)]^2 \\ &\quad + \frac{1}{2n+1}(z^2-1)\left[P_n(z)\frac{\mathrm{d}R_n(z)}{\mathrm{d}z} - R_n(z)\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}\right] + C.\end{aligned} \tag{6.28}$$

Next, if use is made of the decomposition (5.93) and the identity (5.126), equation (6.28) becomes

$$\int \mathrm{d}z\, [P_n(z)]^2 = \frac{1}{2n+1}z[P_n(z)]^2 + \frac{1}{2n+1}(z^2-1)\left[P_n(z)\frac{\mathrm{d}V_n(z)}{\mathrm{d}z} - V_n(z)\frac{\mathrm{d}P_n(z)}{\mathrm{d}z}\right] + C. \tag{6.29}$$

The right-hand side of the above result may still be simplified if one observes that the derivatives in the square bracket may be eliminated with the aid of formulae (5.106) and (3.14); this yields

$$\begin{aligned}\int \mathrm{d}z\, [P_n(z)]^2 &= \frac{1}{2n+1}P_n(z)[zP_n(z) - P_{n-1}(z)] \\ &\quad + \frac{n}{2n+1}[P_{n-1}(z)V_n(z) - V_{n-1}(z)P_n(z)] + C.\end{aligned} \tag{6.30}$$

Then, if one exploits the identity (5.111), equation (6.30) is cast into the form

$$\int \mathrm{d}z\,[P_n(z)]^2 = \frac{1}{2n+1} z[P_n(z)]^2 + \frac{2}{2n+1} \sum_{k=1}^{n-1} P_k(z)[zP_k(z) - P_{k+1}(z)] + C. \tag{6.31}$$

This formula was given, however without a proof, by Whittaker and Watson [30, p 330]. Since it holds that

$$\sum_{k=1}^{n-1} P_k(z)[zP_k(z) - P_{k+1}(z)] = \sum_{k=1}^{n-1} \frac{k}{2k+1} P_{k-1}(z)P_k(z) - \sum_{k=1}^{n-1} \frac{k}{2k+1} P_k(z)P_{k+1}(z)$$
$$= -\frac{n-1}{2n-1} P_{n-1}(z)P_n(z) + \sum_{k=1}^{n-1} \frac{1}{4k^2-1} P_{k-1}(z)P_k(z) \tag{6.32}$$

(the first equality in the above equation results from manipulating with the summand at the extreme left with the aid of the relation (3.9)), equation (6.31) may be further transformed yielding

$$\int \mathrm{d}z\,[P_n(z)]^2 = \frac{1}{2n+1} z[P_n(z)]^2 - \frac{2(n-1)}{4n^2-1} P_{n-1}(z)P_n(z)$$
$$+ \frac{2}{2n+1} \sum_{k=1}^{n-1} \frac{1}{4k^2-1} P_{k-1}(z)P_k(z) + C \tag{6.33}$$

which coincides with the finding of Hargreaves [36] (cf also [11, p 35]).

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